University of Zurich[UZH]

PHY401
Exercise Sheet 10

HS2025
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Setup In this exercise sheet we will explore the basic properties of two coupled superconductors. The order parameter ψ of each superconductor can be written as

$$\psi = |\psi| e^{i\phi} \equiv \sqrt{\rho_s} e^{i\phi} \tag{1}$$

where ρ_s is interpreted as the density of Cooper pairs (two-electron pairs) and ϕ is the phase of the order parameter. This order parameter behaves as a wave function and satisfies the time-dependent Schrödinger equation

$$i\dot{\psi} = H\psi$$

where we set $\hbar = 1$ for simplicity, $\dot{\psi} \equiv \partial\psi/\partial t$, and H is the Hamiltonian.

1 1 DC Josephson effect ★ New

If the two superconductors are separated, they evolve independently. The wave function of the two-superconductor system can be written as a vector $\Psi = (\psi_1, \psi_2)$, where ψ_i is the order parameter of the i-th superconductor. The Schrödinger equation for the two system is given by

$$i \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Now we bring the two superconductors closer and put a thin insulating layer between the two superconductors, which creates a coupling effect. As we discussed in exercise sheet 7, the coupling is characterized by the off-diagonal elements in the Hamiltonian. Let us add in an extra coupling constant κ to the Hamiltonian matrix:

$$i \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} = \begin{pmatrix} E_1 & \kappa \\ \kappa & E_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2}$$

This off-diagonal term also encodes the tunneling effect.[1]

(a) Using Eq. (1), prove

$$\frac{\dot{\psi}_i}{\psi_i} = \frac{1}{2}\frac{\dot{\rho}_i}{\rho_i} + i\dot{\phi}_i, \quad i = 1, 2 \tag{3}$$

(b) Rewrite Eq. (2) and prove

$$i\frac{\dot{\psi}_1}{\psi_1} = E_1 + \kappa\sqrt{\frac{\rho_2}{\rho_1}} e^{i\delta}, \quad \delta \equiv \phi_2 - \phi_1 \tag{4}$$

similar for ψ_2.

[1] It is good to review the concept of coupling and the physical interpretation of the off-diagonal term in the Hamiltonian.

(c) Combine Eq. (3) and Eq. (4). Prove that from the real part of the equation, we have

$$-\dot{\phi}_1 = E_1 + \kappa\sqrt{\frac{\rho_2}{\rho_1}}\cos\delta \tag{5}$$

and from the imaginary part, we have

$$\dot{\rho}_1 = 2\kappa\sqrt{\rho_1\rho_2}\sin\delta \tag{6}$$

Similar for ψ_2 and ρ_2.

(d) Use Eq. (6) for ρ_1 and ρ_2 to prove that the Cooper pair density is conserved in the entire system, i.e. $\rho_1 + \rho_2 = \text{const.}$

(e) Prove that the current density is[2]

$$J = 4e\kappa\sqrt{\rho_1\rho_2}\sin\delta \equiv J_c\sin\delta \tag{7}$$

This means that as long as there is a phase difference, there is a current flowing between the two superconductors.

2 2 AC Josephson effect ★ New

Now we assume that the two superconductors have the same properties, i.e. same material, same transition temperature, and same Cooper pair density: $\rho_1 = \rho_2$.

(a) Use Eq. (2) to show that

$$\dot{\delta}(t) = E_1 - E_2 \equiv 2eV, \tag{8}$$

where V is the voltage across the junction. The extra factor of 2 again comes from the fact that each Cooper pair carries 2 electrons. Prove that this further leads to

$$J = J_c\sin(2eVt) \tag{9}$$

(b) Analyze the dimension, and resume the $\hbar$ in Eq. 9.

(c) Calculate the frequency of the AC Josephson effect when the voltage $V = 1\,\mathrm{V}$. How is this compared to the frequency of (1) the visible light, soft X-ray, microwave; (2) plasma frequency; (3) electron cyclotron frequency; (4) phonons?

3 3 SQUID ★ New

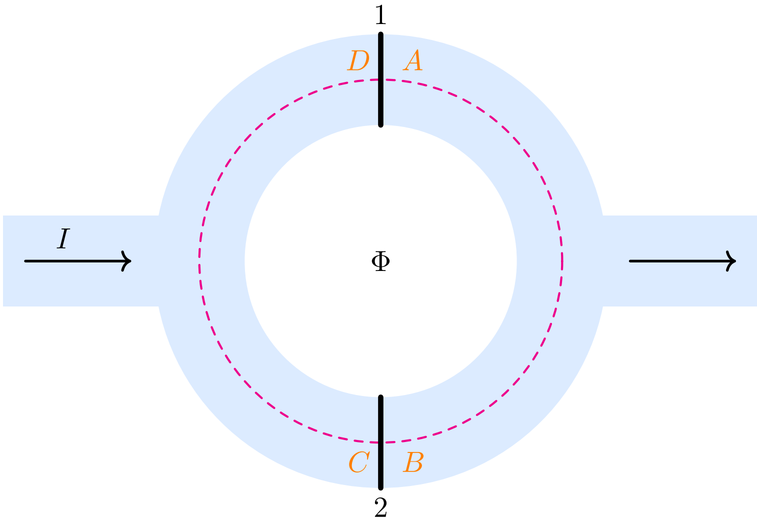
In exercise sheet 9, we derive the London equation when the order parameter ψ is uniform. However, if the phase is not uniform, the current density in the superconductor will not only depend on the vector potential A, but also depend on the gradient of the phase $\nabla\phi$:

$$J = \frac{\rho e}{m}\left(\nabla\phi - 2eA\right) \tag{10}$$

where m is the mass of the electron, and $\hbar = 1$.

Let us consider a superconducting ring with two junctions denoted by 1 and 2, shown in the figure below.

[2] The extra factor of 2 comes from the fact that each Cooper pair carries 2 electrons.



(a) Choose a path deep inside the ring (dashed line) where the current density is zero. Calculate the loop integral of both sides of Eq. (10). Prove that

$$2\pi\frac{\Phi}{\Phi_0} = \delta_1 - \delta_2, \tag{11}$$

where Φ is the magnetic flux, $\Phi_0 = \frac{h}{2e}$ is the flux quantum, $\delta_1 \equiv \phi_A - \phi_D$ and $\delta_2 \equiv \phi_B - \phi_C$.

(b) Assume that two junctions are identical, for simplicity. The current density flowing through the junction i is denoted by J_i. Use the relation ($J = J_c \sin\delta_1 + J_c \sin\delta_2$) to prove that[3]

$$J = 2J_c \sin\left(\frac{\delta_1+\delta_2}{2}\right)\cos\left(\frac{\delta_1-\delta_2}{2}\right) \tag{12}$$

which further leads to

$$J = 2J_c \sin\left(\frac{\delta_1+\delta_2}{2}\right)\cos\left(\frac{\pi\Phi}{\Phi_0}\right)$$

(c) Depending on different circuit setup, δ_1 and δ_2 could evolves differently with time. In some cases, $\delta_1+\delta_2$ leads to a high-frequency oscillation, while $\delta_1-\delta_2$ gives a low-frequency oscillation. Therefore, the current density in Eq. 12 represents a modulated oscillation (high-frequency oscillation modulated by a low-frequency oscillation). Plot the function

$$f(t) = \sin(100t)\cos(5t)$$

and describe what it looks like.

[3]Hint:

$$\delta_1 = \frac{\delta_1+\delta_2}{2} + \frac{\delta_1-\delta_2}{2}, \quad \delta_2 = \frac{\delta_1+\delta_2}{2} - \frac{\delta_1-\delta_2}{2}$$