In this week's exercise, we set $\hbar = 1$ for convenience.

Notation. We use hats for operators and bold for vectors. The spin operators at site i are $\hat{\boldsymbol{S}}_i = (\hat{S}_i^x, \hat{S}_i^y, \hat{S}_i^z)$. The symbol S (no hat, not bold) denotes the *spin quantum number*. When we take the classical approximation later, we set $\hat{\boldsymbol{S}}_i \to \boldsymbol{S}_i = S\,\boldsymbol{n}_i$ with $|\boldsymbol{n}_i| = 1$ and then drop the hats.

1 Ferromagnet ★ New

We take the ferromagnetic Heisenberg Hamiltonian on a 1D lattice of N sites with spacing a:

$$\hat{H} \;=\; -J\sum_{i=1}^{N} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1}, \qquad J > 0, \tag{1}$$

where the index i represents the i-th site, with periodic boundary conditions $\hat{\boldsymbol{S}}_{N+1} \equiv \hat{\boldsymbol{S}}_1$. Let S be the spin quantum number at each site. For electrons $S = \frac{1}{2}$.

(a) *A recap of spin operators*: For a spin-1/2 system, we have the following representation of the spin operators:

$$\hat{S}^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

What are the eigenstates of the operator $\hat{S}^z$?[1] Let us denote these two eigenstates as $|\uparrow\rangle$ and $|\downarrow\rangle$. Calculate:

$$\hat{S}^x\,|\uparrow\rangle\,, \quad \hat{S}^y\,|\uparrow\rangle\,, \quad \hat{S}^x\,|\downarrow\rangle\,, \quad \hat{S}^y\,|\downarrow\rangle$$

and express the results in terms of the two states $|\downarrow\rangle$ and $|\uparrow\rangle$.

(b) Intuitively, the system has the lowest energy when all spins align. We use Dirac notation to denote this state:

$$|\phi\rangle \equiv |\uparrow_1\uparrow_2 \cdots \uparrow_N\rangle \tag{2}$$

where $\uparrow_i$ means the i-th spin is pointing up, i.e. the up-spin eigenstate of the $\hat{S}_i^z$ operator. Prove that this state $|\phi\rangle$ is indeed an eigenstate of the Hamiltonian (Eq. 1).[2]

[1] Hint: An eigenstate $|\psi\rangle$ of an operator $\hat{O}$ satisfies the equation $\hat{O}\,|\psi\rangle = \lambda\,|\psi\rangle$, where λ is the eigen value of the state. In this case, you need to find the states that satisfy $\hat{S}^z\,|\psi\rangle = \lambda\,|\psi\rangle$. You should expect to find two eigenstates.

[2] Hint: To simplify the problem, you could try $N = 2$ first, then extend it to $N = 3$. Again, to prove that $|\phi\rangle$ is indeed an eigenstate of the Hamiltonian, you need to prove that $\hat{H}\,|\phi\rangle = \lambda\,|\phi\rangle$

2 Spin wave in ferromagnet ★ New

Take the Hamiltonian from the last exercise.

(a) The equation of motion of this system is

$$\dot{\hat{\boldsymbol{S}}}_i = J\hat{\boldsymbol{S}}_i \times \left(\hat{\boldsymbol{S}}_{i-1} + \hat{\boldsymbol{S}}_{i+1}\right), \tag{3}$$

where $\dot{\hat{\boldsymbol{S}}}$ is the time derivative of the operator $\hat{\boldsymbol{S}}$. This is a vector form of the equation of motion. Rewrite the equation in terms of $\hat{\boldsymbol{S}}$'s three components: $\hat{S}^x$, $\hat{S}^y$, and $\hat{S}^z$.

(b) We take a classical approximation, we treat the $\boldsymbol{S}$ as a classical variable, meaning that it is not a operator anymore. Each $\boldsymbol{S}$ can be simply written as a 3-dimensional vector (for example, $(0.1, 0.1, 0.2)$). Let's assume the system is sitting at its ground state where all spins are pointing in the z direction, i.e. $S_i^z = S$, and $S_i^x = S_i^y = 0$. A small perturbation will bring the state slightly away from this ground state. Try to argue, in a hand-wavy way, that a small transverse perturbation leads to

$$S^z \approx S, \quad S^x = \delta S^x, \quad S^y = \delta S^y,$$

up to first order, where $\delta S^{x,y}$ is a small value compared to S. This means that z component is almost unchanged under small perturbation. Therefore, we can treat it as a constant.

(c) Assuming a small perturbation is applied. Simplify the equation of motion Eq.2. You should get

$$\delta\dot{S}_i^x = -JS\left[\left(\delta S_{i+1}^y - \delta S_i^y\right) - \left(\delta S_i^y - \delta S_{i-1}^y\right)\right], \tag{4}$$

and similar expression for the y component. In fact, we can actually discard the δ sign in front of the S, because effectively $\delta S^x = S^x$, and $\delta S^y = S^y$.

(d) Instead of discretizing the system like we usually do in physics, we will do the opposite this time: let us "continu-nize" the system. This means that the lattice constant is so small that we can treat the system as a continuous system. Therefore, the spin operator now becomes a function of position, $S_i \Rightarrow S(x)$, where x is the coordinate. Prove that, with this approximation, the equation can be rewritten as

$$\frac{\partial S^x}{\partial t} = -JSa^2\frac{\partial^2 S^y}{\partial x^2}, \quad \frac{\partial S^y}{\partial t} = JSa^2\frac{\partial^2 S^x}{\partial x^2}, \tag{5}$$

where a is the lattice constant.

(e) Fourier transform the Eq. 5 to both frequency (energy) and k (momentum) space.[3] You should get the following equation:

$$\mathrm{i}\omega\tilde{S}^x(\omega, k) = -JSa^2k^2\tilde{S}^y(\omega, k) \tag{6}$$

and similar expression for the y component.

(f) Prove that this system has a dispersion of the form

$$\omega \propto (ka)^2$$

Explain why this LOOKS different from what you learned in the lecture, where $\omega \propto (1-\cos(ka))$.

3 Magnetic domain walls ❖ From 2024

Hidden due to University of Zurich policy.

[3] Use $f(x,t) = \int \frac{dk}{2\pi}\frac{d\omega}{2\pi}\,\tilde{f}(\omega,k)\,e^{i(kx-\omega t)}$. Then $\partial_t \to -i\omega$, $\partial_x^2 \to -k^2$.

4 (Optional) We restore the $\hbar$ in this exercise. Spin waves in a two-dimensional antiferromagnet ❖ From 2024

Hidden due to University of Zurich policy.