Due on 14 Oct. 2025 Discussion on 14 Oct. 2025

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1 From second to first order transitions in Landau theory ★ New

Setup. Consider a uniform scalar order parameter m (e.g. magnetization) with $m \mapsto -m$ symmetry. The Landau free-energy density is

$$f(m; T, g, h) = f_0 + \frac{a(T)}{2} m^2 + \frac{b(g)}{4} m^4 + \frac{c}{6} m^6 - h\,m,$$

with $c \geq 0$ for stability, $a(T) = a_0\,(T - T_c)$ $(a_0 > 0)$, and a non-thermal control parameter g that can tune $b(g)$. Unless stated otherwise, set $h = 0$.[1]

(a) Continuous (second-order) transition. Before doing the exercise, you can try out different combination of positive/negative a, b, and c and plot out the function $f(m)$. It is worth getting some feels of how the shape of the function changes with different signs of a, b, c. For all sub-exercises (a), we assume $c = 0$ for now.

(a) **Symmetry constraint.** *Prove* that odd powers of m are forbidden at $h = 0$ by the $m \to -m$ symmetry, so that the lowest non-trivial terms are m^2, m^4, m^6.

(b) **Equilibrium order parameter and β.** Assume $b > 0$. *Prove* that the global minimum satisfies

$$m_{\text{eq}}(T) = \begin{cases} 0, & T > T_c, \\ \pm\sqrt{-\,a(T)/b}, & T < T_c, \end{cases}$$

and hence the order-parameter critical exponent is $\beta = \frac{1}{2}$.[2]

(c) **Continuity of f and specific-heat jump.** Define $f_{\min}(T) = \min_m f(m; T, g, 0)$.[3] *Prove* that $f_{\min}(T)$ is continuous at T_c, while the specific heat $C = -T\,\partial^2 f_{\min}/\partial T^2$ has a finite jump at T_c (no divergence). *Hint:* Evaluate $f_{\min}$ below T_c by inserting m_{eq} from (a2).

(d) (Optional: finish everything else before coming back for this)
Susceptibility and γ. Turn on a uniform field h (keep $b > 0$). *Prove* for $T > T_c$ that the linear susceptibility $\chi = \partial m/\partial h|_{h=0}$ obeys $\chi = 1/a(T)$, and infer $\gamma = 1$. *Prove* that the same exponent holds for $T < T_c$ when the response is computed around $m_{\text{eq}} \neq 0$. [4]

(e) (Optional: finish everything else before coming back for this)
Critical isotherm and δ. At $T = T_c$ and small h, *prove* that $m \propto h^{1/3}$, hence $\delta = 3$.[5]

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[1] It is worth reminding yourself that only a is T-dependent.

[2] β characterizes how the order parameter vanishes as the transition is approached from below: $m \sim (-t)^{\beta}$ for $t \to 0^-$ at $h = 0$ (equivalently $m \sim (-a)^{\beta}$ since $a \propto t$).

[3] Here $\min_m f$ means the *global* minimum of $f(m)$ over all real m. Practically: solve $\partial f/\partial m = 0$ for all stationary points m_i and compare the values $f(m_i)$. The phase realized in equilibrium is the one with the smallest f. If two minima have equal free energy the system is on a phase boundary (coexistence). Other local minima with larger f are *metastable*.

[4] γ governs the divergence of the (isothermal) susceptibility: $\chi \equiv \partial m/\partial h|_{h=0} \sim t^{-\gamma}$ as $t \to 0^+$ (and likewise from below when computed about $m_{\text{eq}} \neq 0$).

[5] δ is defined by the *critical isotherm* at $T = T_c$: $m \sim h^{1/\delta}$ for small h.

(b) Bridging to first order: softening the quartic term. Assume $b = b(g)$ can change sign under variation of the control parameter g (pressure, composition, ...), while $c > 0$ remains fixed. Note that you need to restore the m^6 to the free energy, since you have $c \neq 0$ now.

(a) **Emergence of extra extrema.** For $b < 0$ and small positive a, *prove* that $f(m)$ has multiple stationary points ($m = 0$ and nonzero m) if and only if $b^2 > 4ac$. (Optional) Prove that the $m = 0$ and the largest m are local minima by inspecting $\partial^2 f/\partial m^2$.

(b) **Coexistence condition.** Let $m_\star \neq 0$ denote a nonzero minimum when it exists. Coexistence (first-order transition) occurs when $f(m_\star) = f(0)$.[6] (Optional): Using the stationarity condition for $m_\star$, *prove* that coexistence requires

$$b = -4\sqrt{\frac{a\,c}{3}} \qquad (a > 0, b < 0),$$

You can try to examine the parameter space, i.e. the $a - b$ plane, and mark out the different phases (disordered phase $m = 0$ and ordered phase $m \neq 0$). You will find that the second-order line $a = 0$ ($b > 0$) joins a first-order line through a tricritical point at $(a, b) = (0, 0)$.

(c) **Order-parameter discontinuity on the first-order line.** Does the $m_\star$ undergo a jump across the first-order phase transition? What about the case of second order phase transition? Explain what makes first-order phase transition different than the second order one. For your information, the jump objeys

$$m_\star^2 = \sqrt{\frac{3a}{c}} = -\frac{3b}{4c}$$

(d) (Optional) **Latent heat along the first-order line.** Assume $a(T) = a_0(T - T_c)$ and b, c are T-independent near the transition. With $S = -\partial f_{\min}/\partial T$, *prove* that the latent heat (upon heating across the coexistence line at fixed g) is

$$L = T\left[S_{\text{disordered}} - S_{\text{ordered}}\right] = \frac{a_0 T}{2} m_\star^2 > 0,$$

[6] At a first-order transition two phases have equal Gibbs/Helmholtz free-energy density at the same control parameters; the equality $f(m_\star) = f(0)$ is precisely this condition at $h = 0$. The jump in m at coexistence is the order-parameter discontinuity.