University of Zurich[UZH]

PHY401
Exercise Sheet 5

HS2025
Prof. Fabian Natterer
Prof. Marc Janoschek

Due on 21 Oct. 2025 Discussion on 21 Oct. 2025

---

1 From Landau to Ginzburg-Landau theory ★ New

Landau theory describes a homogeneous system with free energy

$$F_L = \frac{a(T)}{2}\eta^2 + \frac{b}{4}\eta^4 \tag{1}$$

where $a(T) \equiv a_0(T - T_c)$, $b > 0$, no linear coupling (magnetic field for example) is included. We ignore the F_0 term for now. The Ginzburg-Landau theory describes an inhomogeneous system with free energy written in the form of an integral

$$F_{GL} = \int d^3x \, \left\{ \frac{a(T)}{2}\eta^2(x) + \frac{b}{4}\eta^4(x) + G\,(\nabla\eta)^2 \right\} \tag{2}$$

We will now try to figure out why there should be a gradient term in the integral.

(a1) **Recap on Landau theory**. Get the equation of state (EOS) of Landau theory by minimizing Eq.1. You should get

$$a\eta + b\eta^3 = 0 \tag{3}$$

(a2) Let us discard the gradient term for now by setting $G = 0$. Find the EOS of Ginzburg-Landau theory Eq.2 by minimizing the integral in Eq.2.[1]

You should get the same result as Eq.3:

$$a\eta(x) + b\eta^3(x) = 0, \quad \forall x$$

This means that without the gradient term, both theories give the same EOS, thus the same result. It tells you that it is the gradient terms that makes the difference. But why does it have to be gradient (you don't have to answer this question)?

(b1) **Recap on Ising model**. Now let us look at the one-dimensional classical[2] Ising model, without the external field.

$$H = -J\sum_i s_i s_{i+1} \tag{4}$$

Prove that the Hamiltonian can be rewritten as

$$H = \frac{J}{2}\sum_i (s_i - s_{i+1})^2 + \text{constant}$$

---

[1] To do this, you should first assume a small variation on the order parameter $\eta(x) \to \eta(x) + \delta\eta(x)$. This variation will leads to a variation on the free energy as well: $F_{GL} \to F_{GL} + \delta F$. Plug this in the integral expression and you will end up with the following form of equation:

$$\delta F = \int d^3x \ \{\text{some expression}\} \ \delta\eta$$

To minimize the free energy, we requires $\{\text{some expression}\} = 0$. This will give you the EOS.

[2] by classical it means that each spin $s_i = \pm 1$ is a variable, not an operator.

where the constant term is independent of the spin configuration.[3]

(b2) Again Continous-nize the system and prove that the Hamiltonian can be written in terms of the derivatives of the spin:

$$H = \frac{Ja_{\text{lat}}}{2} \int dx \left(\frac{\partial s(x)}{\partial x}\right)^2 + \text{constant}$$

where a_{lat} is the lattice constant.

(c1) **Back to Ginzburg-Landau theory.** Now $G \neq 0$. Assuming we are still in one dimension. Then the gradient $\nabla\eta \to \partial\eta/\partial x$. Discretize the integral in Eq.2 and rewrite it in the summation form. Compare this with the Ising model, and interpret your results.

(c2) (Optional) Discretize the 3-dimensional version of Eq.2 and interpret your results.

(c3) Explain why this gradient term in the Ginzburg-Landau theory makes a difference in the EOS compared to the Landau theory.

(d) (**Optional**) Derive the EOS of Eq.2, with $G \neq 0$. You need to make use of the following formula

$$\int d^3x \ \delta(\nabla f) = \int d^3x \ \nabla(\delta f)$$

2 Correlation length and susceptibility ★ New

There is a continuous spin system in d dimension. The susceptibility χ is defined by

$$m(x) = \int d^dx \ \chi(x - x')\, h(x') \tag{5}$$

where $m(x)$ is the magnetization at x, and $h(x')$ is the magnetic field at x'. In an isotropic system, the susceptibility only depends on the distance r between the two points, and thus $\chi(x - x') = \chi(|x - x'|) \equiv \chi(r)$. This formula tells you how a magnetic field at x' can propagates its influence and induce a magnetization at another place x.

(a1) Let $d = 1$, i.e. a one dimensional system. Let us look at a simple case where the magnetic field is a pulse localized at $x = 0$. Therefore, $h \sim \delta(x)$, where $\delta(x)$ is the delta function. What is the magnetization $m(x)$ at x induced by this magnetic field?

(a2) If the total magnetic field is a summation of multiple small pulse fields, say $h(x') = \sum_i^n c_i \delta(x' - x_i)$, where c_i is a constant, and x_i is where the pulse is located. What is the magnetization $m(x)$ induced by this magnetic field? Try $n = 2$ first, i.e. two pulse fields.

(a3) In fact, every function $h(x)$ can be decomposed in a integral form:

$$h(x') = \int dy \ c(y)\, \delta(x' - y)$$

compare this form with the summation form in (a2). What is the magnetization $m(x)$ induced by this magnetic field?

(a4) Assuming now, in a 3-dimensional isotropic system, the susceptibility takes the form of

$$\chi(r) = \frac{c}{r} e^{-\xi^{-1} r} \tag{6}$$

where c is a constant and ξ is called the correlation length. Use again the pulse field $h = \delta(\boldsymbol{r})$ at $r = 0$, $\xi = 10\,\text{Å}$. Calculate the magnetization m_1 at a distance of $r = 10\,\text{Å}$ and m_2 at a further distance $r = 30\,\text{Å}$, in terms of c. How much smaller is m_2 than m_1. Explain why ξ is called *correlation length*.

[3]Hint: $(a - b)^2 = a^2 + b^2 - 2a \cdot b$

(b1) **From real space to k-space**. In a 1-dimensional system, where $d = 1$, The susceptibility take the form of

$$\chi(x) \propto e^{-\xi^{-1}|x|}$$

Fourier transform the susceptibility to k-space. Prove the following results:

$$\tilde{\chi}(k) \propto \frac{1}{\xi^{-2} + k^2}$$

The convention for the Fourier transform: $\tilde{f}(k) \equiv \int dx\, f(x) \exp(ikx)$. You might encounter a term like $e^{ik\cdot\infty}$, you can simply drop it and ignore this term.

(b2) In a 3-dimensional system, where $d = 3$, the susceptibility takes the form of

$$\chi(\boldsymbol{x}) \propto \frac{1}{|\boldsymbol{x}|} e^{-\xi^{-1}|\boldsymbol{x}|}$$

Fourier transform the susceptibility to k-space. Prove the following results[4]:

$$\tilde{\chi}(\boldsymbol{k}) \propto \frac{1}{\xi^{-2} + k^2}$$

(c1) **(Optional) From k-space to real space**. Again in an 1-dimensional isotropic system, the susceptibility takes the following form in k-space:

$$\tilde{\chi}(k) \propto \frac{1}{\xi^{-2} + k^2}$$

Fourier transform it back to real space and prove that you get the same result as in (b1)[5].

(c2) (Optional) Do the same for a 3-dimensional system.

(d) Use the following definition of free energy in Ginzburg-Landau theory (Following the lecture's convention)

$$F_{\mathrm{GL}} \equiv \int d^3x \,\left\{A\eta^2 + B\eta^4 + G(\nabla\eta)^2 - h\eta\right\}$$

The corresponding susceptibility in k-space is (the momentum variable is now q following the notation in the lecture):

$$\tilde{\chi}(q) = \frac{1}{2A + 2Gq^2}$$

Let's now set $G = 0$. Prove that in real space the susceptibility is simply a delta function. This means that the magnetic field affects only locally, and its influence can no longer propagate. Explain why this is the case. What can you say about the correlation length?

---

[4] Hint: in 3 dimension, the Fourier transform is defined as

$$\tilde{f}(\boldsymbol{k}) \equiv \int d^3\boldsymbol{x}\, f(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

there are multiple ways to simplify this expression. (1) We set up a spherical coordinate system and put $\boldsymbol{x}$ to the z axis, and with this system the dot product becomes $\boldsymbol{k}\cdot\boldsymbol{x} = kr\cos(\theta)$. (2) In spherical coordinate system, the integral $\int d^3\boldsymbol{x} = \int r^2 \sin\theta \; d\phi d\theta dr$. Due to the isotropy, we can integrate out the ϕ and thus $\int d^3\boldsymbol{x} = 2\pi \int r^2 \sin\theta \; d\theta dr$

[5] You might need to make use of the Residue theorem:

$$\oint_\gamma f(z) dz = 2\pi i \sum_i \mathrm{Res}_i(f)$$

where $\mathrm{Res}_i(f)$ is the i^{th} residue inside the loop γ