University of Zurich[UZH]

PHY401
Exercise Sheet 8

HS2025
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Due on 11 Nov. 2025 | Discussion on 11 Nov. 2025

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1 Classical and quantum Hall effect ❖ From 2024

Hidden due to University of Zurich policy.

2 Landau gauge in Quantum Hall effect ★ New

Setup A 2D system in the x-y plane with magnetic field $\boldsymbol{B}$ in z direction. The vector potential $\boldsymbol{A}$ is defined by $\boldsymbol{B} \equiv \nabla \times \boldsymbol{A}$. In this exercise we use the Landau gauge: $\boldsymbol{A} \equiv xB\hat{\boldsymbol{y}}$, i.e. the vector potential is pointing in y direction and it increases with x.

(a1) Make a sketch/illustration/schematics of the system showing the magnetic field, vector potential, the coordinate system, possible electric field, and possible electric current.

(a2) The general form of the Hamiltonian in this system (without applying the gauge) is

$$H = \frac{(\hat{\boldsymbol{P}} - e\boldsymbol{A})^2}{2m}$$

Apply the Landau gauge now so that the vector potential can be rewritten in terms of the magnetic field (defined in the last exercise). Prove that the Hamiltonian can be written as

$$H = \frac{1}{2m}\left[\hat{P}_x^2 + \left(\hat{P}_y - eBx\right)^2\right]$$

(a3) In the lecture we made an ansatz: $\psi(x, y) \equiv \chi_k(x)e^{iky}$. This is because the Hamiltonian is independent of y, which means in y direction the particle behaves like a free particle without experiencing any potential or force. Of course in x direction it is a different story due to the dependence on x in the Hamiltonian. Show that the time-Independent Schrodinger equation can be written as

$$\left[\frac{\hat{P}_x^2}{2m} + \frac{m}{2}\omega_c^2\left(x - \frac{\hbar k}{eB}\right)^2\right]\chi_k(x) = E\chi_k(x)$$

where $\omega_c \equiv eB/m$. Explain why this represents a simple harmonic oscillator with a shifted center at $x_0 \equiv \hbar k/(eB)$

(a4) Show that the energy levels are given by

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right),$$

where n is an integer. The energy is independent of k. Therefore, the degeneracy of the energy levels is given by the number of k values. *You can do this either with rigorous math or intuitive reasoning.*

(b1) **We will make an attempt** to understand the equations above. The first question is, how come the momentum in y direction creates a shift in the center of the harmonic oscillator in x direction? Let us consider a classical system where the particle behaves like a solid ball with charge e sitting right at the origin $(0, 0)$. At time zero $t = 0$, we kick off the ball with a velocity in y direction: $\boldsymbol{v}(t = 0) = (0, v)$. A magnetic field B is applied in z direction. Prove that the ball will move in a circular orbit with radius[1]

$$R = \frac{mv}{eB}$$

(b2) Where is the center of the circular orbit? Make a sketch to depict the magnetic field, the velocity vector, the circular orbit, and the center of it. *Now, if we make the following naive substitution*

$$\hbar k \equiv p_y \equiv mv. \tag{1}$$

Prove that the center is

$$\boldsymbol{r_0} = \left(\frac{\hbar k}{eB}, 0\right)$$

This coincides with our result in exercise (a)

(b3) So it looks like we should be satisfied with this interpretation: The shifted center x_0 is proportional to the *momentum* $\hbar k$ in y direction because the larger the momentum, the larger the circular orbit will be, and thus the further the center. But hold on, what about the energy? Calculate the kinetic energy of the particle with velocity $v \equiv \hbar k/m$ according to Eq. 1. Prove that the energy depends on k.

(b4) But how come in the exercise (a) the energy

$$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right),$$

depends only on n but not on k? Is this conflicting? Another mysterious thing is: classically the electron goes in a circle, but with Landau gauge the electron behaves like a free particle in y direction. How come?

Please take a moment to think before moving on. You are encouraged to draw a sketch to help you visualize the system.

Starting from here, all exercises below are optional.

(c1) **Now we make a second attempt.** Would be great if you could make a recap on the canonical momentum and the mechanical momentum in Hamiltonian mechanics, and a bit of the gauge in electromagnetism. You could use the prompt below and put it in ChatGPT.

> In Hamiltonian mechanics, what is the difference between canonical momentum and mechanical momentum? You could use "electron in a magnetic field" as an example. The Hamiltonian is
>
> `H = (P - eA)^2/2m`
>
> Would be great if you could mention gauge invariance. Please think before you answer. Explain the concept like I am five.

[1] Hint: Use Newton's second law and the definition of the Lorentz force.

(c2) In a classical system, the momentum is a pure function. Use Hamiltonian equation[2] to prove that

$$\underbrace{m\boldsymbol{v}}_{\text{Mechanical Momentum}} = \underbrace{\boldsymbol{P}}_{\text{Canonical Momentum}} - \underbrace{e\boldsymbol{A}}_{\text{Vector Potential}}$$

In fact, this is also true in quantum mechanics. Please use Heisenberg equation get the same result. This is **the mistake** we made in (b). We should not interpret P as mechanical momentum. Therefore, the equations $\hbar k \equiv p_y \equiv mv$ in (b2) are *incorrect*!

(c3) Show that the canonical momentum is gauge dependent, while the mechanical momentum is gauge-invariant.[3] This means that the canonical momentum P is not even an observable.[4] Therefore, we also cannot see k as some kind of wave vector.

(c4) Plug in the Landau gauge and rewrite the above equation only in y direction. From classical considerations, we know that the electron is moving in a circle, therefore the time average of v_y will be zero. Take the time average, and prove

$$x_0 = \frac{\hbar k}{eB}$$

where x_0 is the center of the circular orbit.

(c5) Follow the argument, answer these questions:

(a) Why is the energy independent of k?

(b) Why is the center x_0 proportional to k, but not something like k^2? Could you explain this intuitively?

(c) Does the electron behave like a free particle in y direction?

[2]Hamiltonian equation: $\dot{q} = \frac{\partial H}{\partial p}$

[3]Hint: The Hamiltonian should also be gauge invariant while the vector potential has a gauge degree of freedom.

[4]Similarly, the gauge dependent $\boldsymbol{A}$ is also not an observable, while the gauge-invariant magnetic field $\boldsymbol{B}$ is an observable.