Due on 18 Nov. 2025 Discussion on 18 Nov. 2025

Preface

After this exercise you will be able to construct the Landau theory for a superconductor from sketch (exercise 2) and explain why a persistent current is allowed, and thus zero resistance occurs (exercise 3), using London equation (exercise 1) and Bohr-Sommerfeld quantization (sheet 8).

1 Magnetic field penetration in a superconductor ❖ From 2024

Hidden due to University of Zurich policy.

2 Constructing the Landau Theory for a superconductor ★ New

In this exercise, you will be given a couple of assumptions on a superconductor, and you will be asked to construct the Landau theory based on those assumptions and derive the London equation

Setup We consider a spherical superconductor in a magnetic field with vector potential $\boldsymbol{A}(\boldsymbol{r})$, with a spatially uniform order parameter defined by $\psi(\boldsymbol{r}) = \psi$.

Assumptions

1. *The order parameter of a superconductor is a complex variable $\psi \in \mathbb{C}$.*
2. *The system has a spherical symmetry*
3. *Magnetic field will suppress the superconductivity (more like an observation)*

(a1) We can start with defining the free energy of the superconductor and the magnetic field, separately. Prove that the free energy of the superconductor

$$F_{\text{SC}} = \int d^3r \left(\frac{a(T)}{2} |\psi|^2 + \frac{b}{4} |\psi|^4 \right)$$

and the free energy of the magnetic field

$$F_{\text{mag}} = \int d^3r \left(\frac{1}{2\mu} \boldsymbol{B}^2 \right) \equiv \frac{1}{2\mu} \int d^3r \, (\nabla \times \boldsymbol{A})^2$$

both satisfy the requirements of: (0) the free energy being a scalar; (1) free energy being a real value; (2) the free energy being invariant under gauge transformations; and (3) the free energy being invariant under rotational transformations. [1]

[1] Sidenote: $|\psi|^2 \equiv \psi\psi^*$

(a2) Define the *naive* total free energy (pure summation, without interaction between the superconductor and the magnetic field) as

$$F_{\text{naive}} = F_{\text{SC}} + F_{\text{mag}}$$

Minimize the total free energy by taking the variational derivative with respect to the order parameter ψ and the vector potential $\boldsymbol{A}$ separately.[2][3] And find the equations of state (EOS) for both the order parameter ψ and the vector potential $\boldsymbol{A}$. Argue that they are independent of each other.

(b1) From exercise (a) we know that the naive summation of two free energies does not provide a good description of the system: the superconductor and the magnetic field are independent and they don't affect each other. Therefore, we need to add an *interaction term* that involves both the order parameter ψ and the vector potential $\boldsymbol{A}$ to the total free energy.

Here is a list of possible interaction terms:

$$\psi\boldsymbol{A}, \quad \psi\boldsymbol{e}_z\cdot\boldsymbol{A}, \quad \psi\psi^*\boldsymbol{A}\cdot\boldsymbol{e}_z, \quad \psi\psi^*\boldsymbol{A}\cdot\boldsymbol{A},$$

where $\boldsymbol{e}_z$ is the unit vector in the z-direction. Take a moment to think. Which one of these is the most reasonable interaction term[4]?

(b2) We can add in the interaction term now and the total free energy becomes

$$\begin{aligned} F_{\text{total}} &= F_{\text{SC}} + F_{\text{int}} + F_{\text{mag}} \\ &= \int d^3r \underbrace{\left(\frac{a(T)}{2}|\psi|^2 + \frac{b}{4}|\psi|^4\right)}_{\text{superconductor}} + \underbrace{\frac{c}{2}|\psi|^2|\boldsymbol{A}|^2}_{\text{interaction}} + \underbrace{\left(\frac{1}{2}\left(\nabla\times\boldsymbol{A}\right)^2\right)}_{\text{magnetic field}} \end{aligned} \tag{1}$$

where we set $\mu = 1$ for convenience, and this can be done by rescaling the free energy, which does not affect the physics at all. Argue that the coefficient c needs to be positive, i.e. $c > 0$.[5]

(b3) Derive the EOS by taking the variational derivative with respect to $\boldsymbol{A}$, and prove:

$$c\boldsymbol{A}|\psi|^2 + \nabla\times\boldsymbol{B} = 0.$$

And then use Maxwell's equations to further show

$$\boldsymbol{j} = -c|\psi|^2\boldsymbol{A}. \tag{2}$$

This is nothing but the London equation in exercise 1. Comparing the two expressions gives $c = e^2/m$ and $|\psi|^2 = n_s$.

3 Persistent current in a superconductor ★ New

In this exercise, we will show that a persistent current can exist in a superconductor (and thus zero resistance), making use of the London equation and the Bohr–Sommerfeld quantization scheme. Let us consider a cylinder threaded by a magnetic field, and the thickness of this cylinder is much larger than the penetration depth λ. The figure below is a top-down view of the cylinder.

[2] Hint: Treat ψ and ψ^* as independent variables. Therefore, when you do the variational derivative wrt. ψ, the other variable ψ^* is considered a constant.

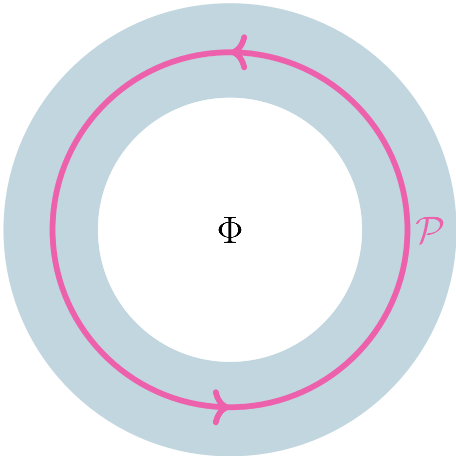
[3] Hint: If you don't want to do a bunch of vector calculus, you can make use of this identity:

$$\frac{\delta F_{\text{mag}}}{\delta\boldsymbol{A}} = 0 \qquad \Rightarrow \qquad \frac{\partial f_{\text{mag}}}{\partial\boldsymbol{A}} + \nabla\times\left(\frac{\partial f_{\text{mag}}}{\partial(\nabla\times\boldsymbol{A})}\right) = 0$$

where f_{mag} is the integrand of the magnetic field free energy. $\boldsymbol{A}$ and $(\nabla\times\boldsymbol{A})$ are treated as independent variables.

[4] Hint: Think about the symmetry and the assumptions we mentioned above

[5] Hint: Two perspectives can be considered: (1) The system always tries to minimize the free energy; (2) the coefficient $a(T)$ controls the superconductivity phase transition.



(a) The Bohr-Sommerfeld quantization scheme states that

$$\oint_{\mathcal{P}} \boldsymbol{p} \cdot d\boldsymbol{r} = 2\pi\hbar n, \quad n \in \mathbb{Z}$$

where $\boldsymbol{p}$ is the canonical momentum, n is an integer, and $\mathcal{P}$ is the circular integration path depicted in the figure above. Let us assume that the path is deep inside the superconductor, and thus there is no current along the path, according to exercise 1. Recall the definition of the canonical momentum, and prove that the magnetic flux through the cylinder is quantized in the following way[6][7]:

$$\Phi = \frac{h}{e} n, \quad n \in \mathbb{Z}$$

(b) Exercise (a) tells us that the magnetic flux is quantized and thus the magnetic field cannot change continuously, given that there is no external field. Since the magnetic field/flux can only be maintained by a loop current,

   (a) Can you explain why there has to be a persistent current in this superconductor?

   (b) Where is the current located?

[6]Hint: You might need to use the Stokes' theorem:

$$\oint_{\mathcal{P}} \boldsymbol{A} \cdot d\boldsymbol{r} = \int_{\mathcal{S}} (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S}$$

[7]More rigrious derivation gives an extra $1/2$ prefactor: $\Phi = \frac{h}{2e} n$