Tutorial: Sheet 9 – **Gauge Mystery**

Xunyang Hong

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Would be great if you could place this note and the exercise sheet 9 side by side.

The suspicious London equation

In exercise (b3) in sheet 9, we derived the London equation from the free energy:

$$\boldsymbol{j} = -c|\psi|^2 \boldsymbol{A}$$

where j is the current density, c is a constant, ψ is the order parameter, and $\boldsymbol{A}$ is the vector potential. A question was raised: The LHS of this equation is an observable, while the RHS is gauge-dependent, i.e. a change in the gauge will result in a change in the current, which does not make sense. How can it be possible? In this section, we will explore this gauge mystery and reconcile the two seemingly contradictory statements.

If you go through the exercise sheet 9, you might notice something suspecious:

- Wiki says that the (second) London equation is

  $$\nabla \times \boldsymbol{j} \sim -|\psi|^2 \boldsymbol{B} \equiv -|\psi|^2 (\nabla \times \boldsymbol{A})$$

  where we drop the unimportant coefficients. This equation does not necessarily imply $\boldsymbol{j} \sim |\psi|^2 \boldsymbol{A}$. Only with a proper gauge choice, we can have $\boldsymbol{j} \sim |\psi|^2 \boldsymbol{A}$.

- But then how come we end up with the equation (5) in sheet 9? If you again take a look at the free energy in Eq. (5), you will see that the free energy is already gauge-dependent, more specifically, the interaction term in the free energy:

  $$f_{int} = |\psi|^2 |\boldsymbol{A}|^2$$

  If you change $A \to A + \nabla\chi$, the free energy also changes:

  $$f_{int} \to |\psi|^2 \left(\boldsymbol{A} + \nabla\chi\right)^2$$

  This means that the entire free energy depends on gauge choice. As free energy should be an observable. The fact that it is gauge dependent doesn't make sense, unless you already choose a gauge before you write down the free energy.

  If we did implicitly choose a gauge before writing down the free energy, then the equation $\boldsymbol{j} \sim |\psi|^2 \boldsymbol{A}$ would not be too surprisingly. This is because if the gauge is already fixed, both sides are also fixed. No gauge problem any more.

- But what is the gauge? And where exactly did we impose that gauge? If you take a step back and review the statement of the setup in exercise 2: *We consider a superconductor ..., with a spatially uniform order parameter* $\psi(\boldsymbol{r}) = \psi$. The question is, can it really be uniform in a magnetic field? To answer this question, let us look at the equation of state (EOS) for ψ :

  $$\left(\frac{a}{2} + \frac{c}{2}|\boldsymbol{A}(\boldsymbol{r})|^2\right)\psi + \frac{b}{2}|\psi|^2\psi = 0$$

  which further leads to

  $$|\psi|^2 = -\frac{\left(a + c|\boldsymbol{A}(\boldsymbol{r})|^2\right)}{b}$$

Depending on the gauge choice, the RHS can depend on position $\boldsymbol{r}$. To enforce the uniformity of ψ, the gauge choice should be such that the $\boldsymbol{A}(\boldsymbol{r})$ is also uniform. This is possible, since *deep* inside the superconductor, the magnetic field B is zero, and thus the vector potential can be naturally a constant, or preferably a zero. This is the necessary condition for the gauge we should impose to get to the London equation in sheet 9.

- Of course, this doesn't fully answer the question. What should be the free energy without fixing the gauge? How do we keep the gauge freendom throughout the entire process explictly, without imposing any additional constraint on gauge? What would happen if it is near the surface of the superconductor? Don't worry, those questions will be answered once you have a well-designed, gauge-invariant free energy.

Constructing a proper Free energy

Let us write down the normal Ginzburg-Landau theory with proper $\boldsymbol{r}$ dependence for a superconductor without the magnetic field first:

$$F = \int d^3r \left(\frac{a}{2}|\psi(\boldsymbol{r})|^2 + \frac{b}{4}|\psi(\boldsymbol{r})|^4 + \frac{c}{2}|\nabla\psi(\boldsymbol{r})|^2 \right)$$

If we want to add the interaction term with magnetic field to this free energy, it has to fulfill two requirements

1. The integrand f should be gauge-invariant.
2. When ψ is uniform, it reduces to the Eq. (5) in the sheet 9.

Before we move on, it will be beneficial to rewrite ψ by its magnitude and its phase: $\psi(\boldsymbol{r}) \equiv |\psi(\boldsymbol{r})|e^{i\phi(\boldsymbol{r})}$. The term that matters the most is the gradient term in the free energy, as all the other terms are "phase-insensitive".

$$\nabla\psi(r) \equiv \nabla\left(|\psi|e^{i\phi(r)}\right) = \left[\nabla|\psi| + \underbrace{|\psi|i\nabla\phi(r)}_{extra}\right] e^{i\phi(r)}$$

the extra term comes from the phase of the order parameter. In principle, since the supercharge density $n_s \sim |\psi|^2$, the phase gets canceled out. This means that the phase would not affect the observable supercharge density. This sounds a bit similar to the gauge of the magnetic field: the gauge does not affect the observable magnetic field B.

Therefore, if we try to combine the magnetic field and the phase term in a good way, and somehow when you change the gauge, the extra phase term $i\nabla\phi$ and the extra gauge term $\nabla\chi$ cancel out each other, then we would end up with a perfect gauge-invariant free energy! Actually, it is quite promising, since those two terms have very similar forms! Why don't we just add an extra term $iA\psi$ in addition to the gradient term:

$$\nabla\psi - i\boldsymbol{A}\psi \equiv \left[\nabla|\psi| + |\psi|i\nabla\phi(r) - i\boldsymbol{A}|\psi|\right]e^{i\phi(r)}$$

and enforce a *gauge transform* to the *system*:

$$\boldsymbol{A}(r) \rightarrow \boldsymbol{A}(r) + \nabla\chi(r), \quad \psi(r) \rightarrow \psi(r)e^{i\chi(r)}$$

Or effectively, the phase changes $\phi(r) \rightarrow \phi(r) + \chi(r)$. Keep in mind that we are now changing the gauge of the entire system, not just the gauge of the magnetic field. the superconductor plus the magnetic field are now considered as a one big system.

Under this gauge transformation, the gradient terms become:

$$\left[\nabla|\psi| + |\psi|i(\nabla\phi + \nabla\chi) - i(\boldsymbol{A} + \nabla\chi)|\psi|\right]e^{i\phi+i\chi} = \left[\nabla|\psi| + |\psi|i\nabla\phi - i\boldsymbol{A}|\psi|\right]e^{i\phi+i\chi} = Unchanged$$

The overall phase $e^{i\phi+i\chi}$ does not matter as they will disapear after taking the module. Nice! We get an interaction term that is gauge invariant! It tells us that the term $\nabla\psi - i\boldsymbol{A}\psi$ is a suitable interaction term that satisfies all requirements: gauge-invariance, real number ... Without further ado, let us add this to the free energy:

$$F = \int d^3r \left(\frac{a}{2}|\psi(\boldsymbol{r})|^2 + \frac{b}{4}|\psi(\boldsymbol{r})|^4 + \frac{c}{2}\left|\nabla\psi(\boldsymbol{r}) - i\boldsymbol{A}(r)\psi(r)\right|^2 \right) + \underbrace{\frac{1}{2}(\nabla\times\boldsymbol{A})^2}_{puremagnetic}$$

You can even rewrite the free energy in a more compact form:

$$F = \int d^3 r \, \left(\frac{a}{2}|\psi|^2 + \frac{b}{4}|\psi|^4 + \frac{c}{2}\left|(-i\nabla - \boldsymbol{A})\psi\right|^2 \right) + F_{magnetic}$$

where $(-i\nabla - A)\psi \equiv -i\nabla\psi - A\psi$. This is the complete free energy for a superconductor. This is also the free energy introduced in the lecture.[1] A quick sanity check shows that when ψ is uniform, $\nabla\psi = 0$, this free energy reduces to Eq. (5) in sheet 9.

Equation of state

Now we can re-examine the equation of state. Again to simplfy the problem, we assume that the $\boldsymbol{r}$ dependence of ψ is only in the phase. The current is now:

$$\boldsymbol{j} = \frac{e\hbar}{m}|\psi|^2 \left\{ \nabla\phi - \frac{e}{\hbar}\boldsymbol{A} \right\}$$

And this is the equation you have in sheet 10. If you implement a gauge transformation, you will see that the extra gauge term in the RHS cancel out each other, leaving the entire RHS gauge-*invariant*, thus the LHS gauge-*invariant*!

If you redefine the vector potential $A' = A - \hbar\nabla\phi/e$, you get the London equation Eq. (1) in sheet 9:

$$\boldsymbol{j} = -\frac{e^2}{m}|\psi|^2 \boldsymbol{A}'$$

This "redefinition" of the vector potential is equivalent to a gauge fixing.

There are a lot more...

Does the term $(-i\hbar\nabla - eA)^2$ look familiar? This is also the momnetum term in the Hamiltonian of an electron in a magnetic field:

$$H = \frac{\left(\hat{P} - eA\right)^2}{2m}, \quad \hat{P} \leftrightarrow -i\hbar\nabla$$

This is not a coincidence. In fact, a lot of the times when you apply a magnetic field to a system, the simple substitution of $\hbar\nabla \to \hbar\nabla - ieA$ would work perfectly. There is even a term for that: *Minimal substitution*. This is the case for a simple electron in a magnetic field: $\hat{P} \to \hat{P} - eA$. This is also the case for the Landau theory for a superconductor: $\hbar\nabla \to \hbar\nabla - ieA$.

Another example is the Dirac field in quantum electrodynamics. To couple the Dirac field with the electromagnetic field, you simply replace the derivative operator ∂_μ with $\partial_\mu + ieA_\mu$.

[1] Note that in this note all unnecessary constants or coefficients are set to 1, or simply dropped. If you want to be super precise, the gauge transformation would be $A \to A + \hbar\chi/e$. And the interaction term would be $|(-i\hbar\nabla - eA)\psi|^2$

[2] With all coeffcients resumed properly